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ON DUALISTIC CONTRACTIVE MAPPINGS

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Abstract. The purpose of the present paper is to continue the study of the fixed point theory in dual partial metric spaces. We investigate fixed point theorems for Kannan mappings and weakly contractive mappings in dual partial metric spaces. The counterpart theorems provided in the partial metric spaces are retrieved as particular cases of our new results. Moreover, we give some examples to elucidate that the contractive conditions in the statement of our new fixed point theorems can not be replaced by those contractive conditions in the statement of the partial metric counterpart fixed point theorems.

Keywords: fixed point, partial metric, dual partial metric, complete, Kannan mapping, weakly contractive, quasi-metric.

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1. Introduction and preliminaries

Throughout this paper the letters $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, set of real numbers and set of natural numbers, respectively. Let us recall some basic definitions and terminologies to make this paper self-sufficient.
Let X be a nonempty set and let $T : X \to X$ be a self mapping. A point $x \in X$ is called a fixed point of T provided $x = T(x)$. Let $x_0 \in X$, then a simple iterative method defines a sequence $\{x_n\}$ in X by

$$x_n = T(x_{n-1}) \text{ for all } n \in \mathbb{N}.$$

This particular sequence is known as Picard iterative sequence with initial point x_0.
A self-mapping T on a metric space X is said to be a Banach's contraction mapping if there exists $k \in [0, 1[$ such that

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in X.$$

The Banach fixed point theorem can be stated as follows:

Theorem 1.1. *If (X, d) is a complete metric space and $T : X \to X$ is a Banach's contraction self-mapping, then T has a unique fixed point $x^* \in X$. Moreover, the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^*, for every $x \in X$.*

In the literature one can find many generalizations of Banach fixed point theorem. In what follows we focus on two of the mentioned generalizations.
Kannan [6] considered a class of self-mapping $T : X \to X$ satisfying the following condition:

$$d(T(x), T(y)) \leq \frac{k}{2}[d(x, T(x)) + d(y, T(y))] \tag{1}$$

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for all $x, y \in X$ and some $k \in [0,1[$. Nowadays, the self-mappings satisfying condition (1) are called Kannan contraction mappings. Taking into account the new type of contraction, an extension of Banach's fixed point theorem was given by Kannan as follows:

Theorem 1.2. *Let (X,d) be a complete metric space and let $T : X \to X$ be a Kannan contraction mapping, then T has a unique fixed point $x^* \in X$. Moreover, the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^*, for every $x \in X$.*

In [4], Dugundji and Granas provided another generalization of Banach's fixed point theorem. To this end, they consider a new class of self-mappings $T : X \to X$ satisfying the following condition:

$$d\left(f\left(x\right), f\left(y\right)\right) \leq \overline{\alpha}\left(x,y\right) d\left(x,y\right), \tag{2}$$

for all $x, y \in X$, where the function $\overline{\alpha} : X \times X \to [0,1[$ holds, for every $0 < a \leq b$, such that

$$\theta\left(a,b\right) = \sup\left\{\overline{\alpha}\left(x,y\right) : a \leq d\left(x,y\right) \leq b\right\} < 1.$$

According to [4], a self-mapping $T : X \to X$ satisfying condition (2) is said to be weakly contractive. For this new family of self-mappings the generalization of Banach's fixed point theorem given by Dugundji and Granas can be stated as follows:

Theorem 1.3. *Let (X,d) be a complete metric space. If $T : X \to X$ is a weakly contractive mapping, then T has a unique fixed point x^* and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^*, for every $x \in X$.*

In 1994, Matthews [7] introduced the concept of partial metric spaces and he proved a fixed point theorem in such spaces that turned out to be a suitable mathematical tool for program verification. The aforementioned theorem is now known as Matthews' fixed point theorem and it is a generalization of Banach's fixed point theorem for the new class of metric spaces.
Let us recall that a partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_0^+$ satisfying for all $x, y, z \in X$ the following properties:

(1) $x = y \Leftrightarrow p\left(x,x\right) = p\left(x,y\right) = p\left(y,y\right),$
(2) $p\left(x,x\right) \leq p\left(x,y\right),$
(3) $p\left(x,y\right) = p\left(y,x\right),$
(4) $p\left(x,y\right) \leq p\left(x,z\right) + p\left(z,y\right) - p\left(z,z\right).$

Notice that a metric on a set X is a partial metric p such that $p(x,x) = 0$ for all $x \in X$.
As usual a partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.
Matthews [7] shows that each partial metric p on X generates a T_0 topology $\mathcal{T}[p]$ on X whose base is the family of open p-balls $\{B_p\left(x,\varepsilon\right) : x \in X,\ \varepsilon > 0\}$, where

$$B_p\left(x,\varepsilon\right) = \left\{y \in X : p\left(x,y\right) < p\left(x,x\right) + \varepsilon\right\}$$

for all $x \in X$ and $\varepsilon > 0$. It follows that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if $p\left(x,x\right) = \lim_{n\to\infty} p\left(x,x_n\right)$.
According to [7], a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a partial metric space (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p\left(x_n,x_m\right)$ exists and is finite. In addition, a partial metric space (X,p) is said to be complete provided that every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}[p]$, to a point $x \in X$ such that $p\left(x,x\right) = \lim_{n,m\to\infty} p\left(x_n,x_m\right)$.
In [7], Matthews proved that every partial metric p on a nonempty set X induces a metric $p^s : X \times X \to \mathbb{R}_0^+$ by

$$p^s\left(x,y\right) = 2p\left(x,y\right) - p\left(x,x\right) - p\left(y,y\right)$$

for all $x, y \in X$.
In the light of the preceding notion, Matthews proved the next extension of Banach's fixed point theorem.

Theorem 1.4. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a self-mapping such that there exists $\alpha \in [0, 1[$ satisfying*

$$p(T(x), T(y)) \leq \alpha p(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\mathcal{T}[p^s]$, for every $x \in X$.

A large number of mathematicians follow Theorem 1.4 and present a huge number of articles in partial metric spaces (see for example [2, 3, 5, 8, 9, 12, 13]). Inspired by the metric results, Theorem 1.2 and Theorem 1.3, in [1] Alghamdi *et al.* have extended the notions of Kannan mapping and weakly contractive mapping to the context of partial metric spaces (see also [3] for a special type of Kannan mappings in partial metric spaces endowed with an order). Besides, they have yielded partial metric versions of Theorems 1.2 and 1.3. In order to recall the aforementioned versions we introduce the Kannan and weakly contractive self-mappings in the partial metric spaces. Thus, given a partial metric space (X, p), a self-mapping $T : X \to X$ is said to be

(1) Kannan if there exists $k \in [0, 1[$ such that

$$p(T(x), T(y)) \leq \frac{k}{2}\left[p(x, T(x)) + p(y, T(y))\right]$$

for all $x, y \in X$,

(2) weakly contractive if there exists $\alpha : X \times X \to [0, 1[$ such that for every $a, b \in \mathcal{R}$ with $0 \leq a \leq b$

$$\theta(a, b) = \sup\{\alpha(x, y) : a \leq p(x, y) \leq b\} < 1,$$

and for all $x, y \in X$,

$$p(T(x), T(y)) \leq \alpha(x, y)\ p(x, y).$$

From the above notions, the following two extensions of Matthews fixed point theorem were provided in [1].

Theorem 1.5. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a weakly contractive mapping. Then T has a unique fixed point $x^* \in X$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(p^s)$, for every $x \in X$. Moreover, $p(x^*, x^*) = 0$.*

Theorem 1.6. *Let (X, p) be a complete partial metric space and let $T : X \to X$ be a Kannan mapping. Then T has a unique fixed point $x \in X$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(p^s)$, for every $x \in X$. Moreover, $p(x^*, x^*) = 0$.*

Later on, in 1996, Neill introduced the dual partial metric as a generalization of partial metric (pmetric) p in order to expand the connections between partial metrics and semantics via valuation spaces [11]. According to Neill a dual partial metric defined on a nonempty set X is a function $D : X \times X \longrightarrow \mathbb{R}$ such that

$$D(x, y) = p(x, y) - p(x, x) - p(y, y).$$

It is easy to verify that D satisfies the following properties for all $x, y, z \in X$:

(1) $x = y \Longleftrightarrow D(x, x) = D(y, y) = D(x, y)$,

(2) $D(x,x) \leq D(x,y)$,
(3) $D(x,y) = D(y,x)$,
(4) $D(x,z) \leq D(x,y) + D(y,z) - D(y,y)$.

Remark 1.1. We observe that, unlike pmetric case, if D is a dual pmetric then $D(x,y) = 0$ may not implies $x = y$. The self distance $D(x,x)$ referred to as the size or weight of x, is a feature used to describe the amount of information contained in x. The smaller $D(x,x)$ the more defined x is, x being totally defined if $D(x,x) = 0$. It is obvious that if p is a partial metric then D is a dual partial metric but converse is not true. Note that $D(x,x) \leq D(x,y)$, does not imply $p(x,x) \leq p(x,y)$. Nevertheless, the restriction of D to $\mathbb{R}_0^+$, is a partial metric.
If p is partial metric and p^s is a metric defined by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) \text{ for all } x,y \in X,$$

then we have following relation

$$p^s(x,y) = p(x,y) + D(x,y) \text{ for all } x,y \in X.$$

Following [11], each dual partial metric D on X generates a T_0 topology $\mathcal{T}[D]$ on X which has, as a base, the family of D-balls $\{B_D(x,\epsilon) : x \in X, \epsilon > 0\}$ and $B_D(x,\epsilon) = \{y \in X : D(x,y) < \epsilon + D(x,x)\}$. If (X,D) is a dual pmetric space, then the function $d_D : X \times X \to \mathbb{R}_0^+$ defined by

$$d_D(x,y) = D(x,y) - D(x,x),$$

is a quasi metric on X such that $\mathcal{T}[D] = \mathcal{T}[d_D]$ where $B_\epsilon(x;d_D) = \{y \in X | d_D(x,y) < \epsilon\}$. In this case, $d_D^s(x,y) = \max\{d_D(x,y), d_D(y,x)\}$ defines a metric on X, known as induced metric.
The following definition and Lemma describe the convergence criteria established by Oltra *et al.* in [12].

Definition 1.1. [12] *Let (X,D) be a dual partial metric space.*
(1) *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,D) is called a Cauchy sequence if* $\lim_{n,m\to\infty} D(x_n,x_m)$ *exists and is finite.*
(2) *A dual partial metric space (X,D) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}[D]$, to a point $\upsilon \in X$ such that*

$$D(x,x) = \lim_{n,m\to\infty} D(x_n,x_m).$$

Lemma 1.1. [12]
(1) *Every Cauchy sequence in (X,d_D^s) is also a Cauchy sequence in (X,D).*
(2) *A dual partial metric (X,D) is complete if and only if the metric space (X,d_D^s) is complete.*
(3) *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $\upsilon \in X$ with respect to $\mathcal{T}[(d_D^s)]$ if and only if*

$$\lim_{n\to\infty} D(\upsilon,x_n) = D(\upsilon,\upsilon) = \lim_{n\to\infty} D(x_n,x_m).$$

Inspired by the applicability of dual partial metric spaces to program verification, in 2004 Oltra and Valero [12] established a Banach fixed point theorem for dual partial metric spaces in such a way that the Matthews fixed point theorem is obtained as a particular case. The aforesaid result can be stated as follows:

Theorem 1.7. *Let (X,D) be a complete dual partial metric space and let $T : X \to X$ be a self-mapping such that there exists $\alpha \in [0,1[$ satisfying*

$$|D(T(x),T(y))| \leq \alpha |D(x,y)|,$$

for all $x, y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, $p(x^*, x^*) = 0$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(p^s)$, for every $x \in X$.

Motivated by the preceding result, the study of fixed point theory in dual partial metric spaces have been continued by Oltra *et al.* ([12]), Nazam and Arshad [8, 10] who have given some new fixed point theorems.
For the sake of continuity of the work on fixed point theory in dual partial metric spaces, in this paper we prove fixed point theorems for Kannan mappings and weakly contractive mappings in the framework of dual partial metric spaces extending those results provided by Alghamdi *et al.* in [1]. We will show, with the help of examples, that the new results allow to find fixed points of mappings in some cases in which the partial metric spaces cannot be applied.

2. The results

In this section we prove dual partial metric versions of Theorem 1.5 and 1.6. First of all, we focus our investigation in the possibility of obtaining a fixed point theorem for Kannan mappings, in the spite of Theorem 1.5, in the dualistic partial metric spaces. To this end, let us define such type of self-mappings. Given a dual partial metric space (X, D), a self-mapping $T : X \to X$ is said to be Kannan provided that there exists $k \in [0, 1[$ such that

$$|D(T(x), T(y))| \leq \frac{k}{2}\left[|D(x, T(x))| + |D(y, T(y))|\right]$$

for all $x, y \in X$. Observe that from a particular case of the preceding notion, we retrieve the notion of Kannan mapping when the dualistic partial metric is exactly a partial metric. Example 2.1 shows an instance of Kannan mapping in the dual partial metric spaces. Next we present a new fixed point theorem for Kannan mappings in our context.

Theorem 2.1. *Let (X, D) be a complete dual partial metric space and let $T : X \to X$ be a Kannan mapping. Then the following assertions hold:*

1. *for every $x \in X$, the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges with respect to $\tau(d^s_D)$ to a fixed point of T such that $D(x^*, x^*) = 0$,*
2. *T has a unique fixed point in the set $\{x \in X : D(x, x) \geq 0\}$.*

Proof. (1). Let us consider the Picard iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ with initial guess $x_0 \in X$ (i.e., $x_n = T(x_{n-1})$ for all $n \in \mathbb{N}$). Of course, if there exists $n \in \mathbb{N}$ such taht $x_n = x_{n+1} = T(x_n)$ x_n is a fixed point of T. So we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then we have that

$$\begin{aligned}
|D(x_1, x_2)| &= |D(T(x_0), T(x_1))| \leq \frac{k}{2}[|D(x_0, T(x_0))| + |D(x_1, T(x_1))|] \\
&= \frac{k}{2}[|D(x_0, x_1)| + |D(x_1, x_2)|] \\
(1 - \frac{k}{2})|D(x_1, x_2)| &\leq \frac{k}{2}|D(x_0, x_1)| \\
|D(x_1, x_2)| &\leq \lambda|D(x_0, x_1)|, \text{ where } \lambda = \frac{k}{2-k} \text{ and } 0 < \lambda < 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|D(x_2, x_3)| &= |D(T(x_1), T(x_2))| \\
&\leq \frac{k}{2}[|D(x_1, T(x_1))| + |D(x_2, T(x_2))|].
\end{aligned}$$

Thus $|D(x_2,x_3)| \leq \lambda|D(x_1,T(x_1))| \leq \lambda^2|D(x_0,x_1)|$. Continuing in this way we have

$$|D(x_n,x_{n+1})| \leq \lambda^n|D(x_0,x_1)|. \tag{3}$$

Also we can deduce from the contractive condition that

$$|D(x_n,x_n)| \leq k\lambda^{n-1}|D(x_0,x_1)|. \tag{4}$$

To prove that $\{x_n\}$ is Cauchy sequence in (X,D), we will prove that $\{x_n\}$ is a Cauchy sequence in (X,d^s_D). From (3) and (4), we have

$$\begin{aligned} D(x_n,x_{n+1}) - D(x_n,x_n) &\leq |D(x_n,x_{n+1})| + |D(x_n,x_n)| \\ &\leq \lambda^n|D(x_0,x_1)| + k\lambda^{n-1}|D(x_0,x_1)| \\ &\leq \lambda^n(3-k)|D(x_0,x_1)|. \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, for some $\eta \in \mathbb{N}$,

$$D(x_{n+\eta-1},x_{n+\eta}) - D(x_{n+\eta-1},x_{n+\eta-1}) \leq \lambda^{n+\eta-1}(3-k)|D(x_0,x_1)|$$

for all $n \in \mathbb{N}$. Now using the triangular inequality and the preceding inequalities we have that

$$\begin{aligned} D(x_n,x_{n+\eta}) - D(x_n,x_n) &\leq D(x_n,x_{n+1}) + D(x_{n+1}x_{n+2}) + \ldots \\ &+ D(x_{n+\eta-1},x_{n+\eta}) - \sum_{i=0}^{\eta-1} D(x_{n+i},x_{n+i}) \\ &\leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{n+\eta-1})(3-k)|D(x_0,x_1)| \\ &\leq \frac{\lambda^n}{1-\lambda}(3-k)|D(x_0,x_1)|. \end{aligned}$$

Similarly we can calculate that

$$D(x_{n+\eta},x_n) - D(x_{n+\eta},x_{n+\eta}) \leq \frac{\lambda^n}{1-\lambda}(1+k)|D(x_0,x_1)|.$$

Consequently

$$d^s_D(x_n,x_{n+\eta}) \leq 4\frac{\lambda^n}{1-\lambda}|D(x_0,x_1)|.$$

for all $n \in \mathbb{N}$

It follows that $\lim_{n\longrightarrow\infty} d^s_D(x_n,x_{n+\eta}) = 0$ and, thus, that $\{x_n\}$ is Cauchy sequence in (X,d^s_D). Since (X,D) is complete if and only if (X,d^s_D) is complete, therefore, there exists $x^* \in X$ such that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$ *i.e* $\lim_{n\longrightarrow\infty} d^s_D(x_n,x^*) = 0$. Then by Lemma 1.1, we have

$$D(x^*,x^*) = \lim_{n\longrightarrow\infty} D(x_n,x^*) = \lim_{n,m\longrightarrow\infty} D(x_n,x_m). \tag{5}$$

Due to (4) and (5), we have

$$D(x^*,x^*) = \lim_{n\longrightarrow\infty} D(x_n,x^*) = 0.$$

The fact that $0 = D(x^*,x^*)$ yields that $0 \leq D(x^*,T(x^*))$, since $0 = D(x^*,x^*) \leq D(x^*,T(x^*))$. Moreover,

$$\begin{aligned} D(x^*,T(x^*)) &\leq D(x^*,x_n) + D(x_n,T(x^*)) - D(x_n,x_n) \\ &\leq D(x^*,x_n) + |D(x_n,T(x^*))| + |D(x_n,x_n)| \\ &\leq D(x^*,x_n) + \frac{k}{2}[|D(x_{n-1},x_n)| + D(x^*,T(x^*))] + |D(x_n,x_n)|. \end{aligned}$$

for all $n \in \mathbb{N}$. Hence we obtain that

$$(1-\frac{k}{2})D(x^*,T(x^*)) \leq D(x^*,x_n) + \frac{k}{2}|D(x_{n-1},x_n)| + |D(x_n,x_n)|.$$

Since

$$D(x_n,x_{n+1}) \leq \lambda^n(3-k)|D(x_0,x_1)| + D(x_n,x_n)$$

and $\lim_{n\longrightarrow\infty} D(x_n,x_n) = \lim_{n\longrightarrow\infty} D(x_n,x^*) = 0$ we deduce that $\lim_{n\longrightarrow\infty} D(x_n,x_{n+1}) = 0$ and, in addition, that $(1-\frac{k}{2})D(x^*,T(x^*)) \leq 0$.This implies that

$$D(x^*,T(x^*)) = 0 = D(x^*,x^*) = D(T(x^*),T(x^*)) = 0$$

which gives $x^* = T(x^*)$. Hence we have shown that x^* is a fixed point of T with $D(x^*,x^*) = 0$ and $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_D^s)$, for every $x \in X$.

(2) Now we are left to prove the uniqueness of x^* in the set $\{x \in X : D(x,x) \geq 0\}$. For the purpose of contradiction, suppose that y^* is another fixed point of T such that $D(y^*,y^*) \geq 0$. Then

$$\begin{aligned}
|D(x^*,y^*)| &= |D(T(x^*),T(y^*))| \\
&\leq \frac{k}{2}[|D(x^*,T(x^*))| + |D(y^*,T(y^*))|] \\
&= \frac{k}{2}[|D(x^*,x^*)| + |D(y^*,y^*)|] = \frac{k}{2}|D(y^*,y^*)|.
\end{aligned}$$

We distinguish two possible cases:

Case 1. $D(y^*,y^*) = 0$. It follows that $|D(x^*,y^*)| = 0$ and hence, $D(x^*,y^*) = 0$. Thus $D(x^*,y^*) = D(x^*,x^*) = D(y^*,y^*)$ implies $x^* = y^*$.

Case 2. $D(y^*,y^*) > 0$. Since $0 \leq D(x^*,x^*) \leq D(x^*,y^*)$ and $D(y^*,y^*) \leq D(x^*,y^*)$ we have that

$$1 \leq \frac{D(x^*,y^*)}{D(y^*,y^*)} \leq \frac{k}{2},$$

which is a contradiction.

Consequently x^* is the unique fixed point of T in $\{x \in X : D(x,x) \geq 0\}$. □

The following example shows that Theorem 2.1 does not guarantee the uniqueness of fixed point in general.

Example 2.1. *Consider the pair $(X, D_\vee|_X)$, where $X = \{0, \frac{-1}{2}\}$ and $D_\vee|_X$ is the restriction of the dual partial metric $D_\vee$ defined on $\mathbb{R}$. It is clear that $(X, D_\vee|_X)$ is a complete dual partial metric space. Next define the self-mapping $T : X \to X$ by $T(x) = x$ for all $x \in X$. It is not hard to check that*

$$|D_\vee|_X(T(x),T(y))| \leq \frac{1}{2}\left[|D_\vee|_X(x,T(x))| + |D_\vee|_X(y,T(y))|\right]$$

for all $x, y \in X$. Hence we obtain that T is a Kannan mapping. However, T has two fixed points.

Since every partial metric is a dual partial metric D which satisfies $D(x,y) \in \mathbb{R}^+$ for all $x, y \in X$.Thus, Theorem 2.1 is more general than Theorem 1.6.

A natural question that can be raised is, whether the contractive condition in the statement of Theorem 2.1 can be replaced by the contractive condition in Theorem 1.6. The following easy example provides a negative answer to the question.

Example 2.2. *Consider the complete dual partial metric* $(\mathbb{R}, D_\vee)$*. Define the self-mapping* $T_0 : \mathbb{R} \to \mathbb{R}$ *defined by*

$$T_0(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}.$$

It is easy to check that

$$D_\vee(T_0(x), T_0(y)) \leq \frac{1}{2}\left[D_\vee(x, T_0(x)) + D_\vee(y, T_0(y))\right]$$

for all $x, y \in \mathbb{R}$*. However,* T *has not fixed points. Observe that* T *does not hold the contractive condition in the statement of Theorem 2.1. Indeed, note that for all* $k \in [0, 1[$ *we have that*

$$1 = |D_\vee(-1, -1)| = |D_\vee(T_0(0), T_0(0))| > \frac{k}{2}\left[D_\vee(0, T_0(0)) + D(0, T_0(0))\right] = k|0 \vee (-1)| = 0.$$

Next we focus our investigation in the possibility of obtaining a fixed point theorem for weakly contractive Kannan mappings, in the spirit of Theorem 1.6, in the dual partial metric approach. With this aim we extend the notion of weakly contractive mappings to the dual partial metric spaces. Given a dual partial metric space (X, D), a self-mapping $T : X \to X$ is said to be weakly contractive provided that there exists $\alpha : X \times X \to [0, 1[$ such that for every $0 \leq a \leq b$

$$\theta(a, b) = \sup\{\alpha(x, y) : a \leq |D(x, y)| \leq b\} < 1,$$

and for all $x, y \in X$.

$$|D(T(x), T(y))| \leq \alpha(x, y)\,|D(x, y)|.$$

Observe that as a particular case of the preceding notion we retrieve the notion of weakly contractive mapping when the dual partial metric is exactly a partial metric. The next example yields an instance of weakly contractive mapping in the dual partial metric framework.

Example 2.3. *Consider the dual partial metric space* $([-1, 1], D_\vee)$*. Define the self-mapping* $T_3 : X \to X$ *by*

$$T_3(x) = \frac{x^3}{x^2 + 1}$$

for all $x \in X$*. Further, define* $\alpha : [-1, 1] \times [-1, 1] \to [0, 1]$ *by*

$$\alpha(x, y) = \begin{cases} \dfrac{D_\vee(T_3(x), T_3(y))}{D_\vee(x, y)} & \text{if } D_\vee(x, y) \neq 0 \\ 0 & \text{if } D_\vee(x, y) = 0 \end{cases}$$

Observe that $\dfrac{D_\vee(T_3(x), T_3(y))}{D_\vee(x, y)} > 0$ *provided that* $D_\vee(x, y) \neq 0$*. It is not hard to check that* $\alpha(x, y) \leq \frac{1}{2}$ *for all* $x, y \in [-1, 1]$ *and that* $\theta(a, b) < 1$ *for all* $a, b \in \mathbb{R}$ *with* $0 \leq a \leq b$*. Moreover,*

$$|D_\vee(T_3(x), T_3(y))| \leq \alpha(x, y)|D_\vee(x, y)|$$

for all $x, y \in [-1, 1]$*.*

It is known that weakly contractive mappings are Kannan mappings in the metric spaces. So it seems natural to think that what is the relationship between both types of self-mappings in the dual partial metric spaces. On one hand, Example 2.3 provides an instance of weakly contractive mapping which is not Kannan. Indeed, note that there does not exist $k \in [0, 1[$ such that

$$D_\vee(T_3(0), T_3(1)) \leq \frac{k}{2}\left[D_\vee(0, T_3(0)) + D_\vee(1, T_3(1))\right].$$

On the other hand, Example 2.2 gives an instance of Kannan mapping which is not weakly contractive. Indeed, there does not exist $\alpha : [-1,1] \times [-1,1] \to [0,1[$ such that

$$D_\vee(T_0(0), T_0(0)) \leq \alpha(0,0) D_\vee(0,0).$$

Next we prove a fixed point theorem for weakly contractive mapping in the dual context.

Theorem 2.2. *Let (X, D) be a complete dual partial metric space and let $T : X \longrightarrow X$ be a weakly contractive mapping. Then T has a unique fixed point $x^* \in X$ and the Picard iterative sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_D^s)$, for every $x \in X$, . Moreover, $D(x^*, x^*) = 0$.*

Proof. Consider the Picard iterative sequence $\{x_n\}_{n\in\mathbb{N}}$ with initial point $x_0 \in X$ (i.e., $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$). It is clear that if there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$ then x_n is a fixed point of T. So we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then we have that

$$|D(x_n, x_{n+1})| = |D(T(x_{n-1}), T(x_n))| \leq \alpha(x_{n-1}, x_n)|D(x_{n-1}, x_n)| \leq |D(x_{n-1}, x_n)|.$$

This implies that the sequence $\{|D(x_n, x_{n+1})|\}_{n\in\mathbb{N}}$ is decreasing and bounded from below. So it converges to $r \in \mathbb{R}$ with

$$r = \inf_{n\in\mathbb{N}} |D(x_{n-1}, x_n)| \geq 0.$$

We claim that $r = 0$. Otherwise, $r > 0$ and we have

$$0 < r \leq |D(x_n, x_{n+1})| \leq |D(x_{n-1}, x_n)| \leq \ldots \leq |D(x_0, x_1)|$$

implies $0 < r \leq |D(x_0, x_1)|$ and thus, we deduce that

$$\theta = \theta(r, |D(x_0, x_1)|) = \sup\{\alpha(x, y) : \quad r \leq |D(x, y)| \leq |D(x_0, x_1)|\} < 1.$$

It follows that

$$\begin{aligned} r &\leq |D(x_n, x_{n+1})| \\ &\leq \alpha(x_{n-1}, x_n)|D(x_{n-1}, x_n)| \\ &\leq \theta(r, |D(x_0, x_1)|)|D(x_{n-1}, x_n)| \\ &\leq \theta^2 |D(x_{n-2}, x_{n-1})| \leq \ldots \\ &\leq \theta^n(r, |D(x_0, x_1)|)|D(x_0, x_1)|. \end{aligned}$$

Therefore

$$r \leq \lim_{n\to\infty} \theta^n(r, |D(x_0, x_1)|)|D(x_0, x_1)|.$$

This implies that $r \leq 0$. We conclude that $0 < r \leq 0$, which is a contradiction, consequently $r = 0$. Hence,

$$\lim_{n\to\infty} |D(x_n, x_{n+1})| = 0 \text{ implies } \lim_{n\to\infty} D(x_n, x_{n+1}) = 0.$$

Following similar arguments we can show that $\lim_{n\to\infty} D(x_n, x_n) = 0$, since

$$|D(x_n, x_n)| \leq \alpha(x_{n-1}, x_{n-1})|D(x_{n-1}, x_{n-1})|$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{|D(x_n, x_n)|\}_{n\in\mathbb{N}}$ is decreasing and bounded from below. Next we show that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_D^s). It is clear that

$$\begin{aligned} D(x_n, x_{n+1}) - D(x_n, x_n) &\leq \theta^n(0, |D(x_0, x_1)|)|D(x_0, x_1)| + \theta^n(0, |D(x_0, x_0)|)|D(x_0, x_0)|, \\ &\leq \theta^n \left[|D(x_0, x_1)| + |D(x_0, x_0)|\right]. \end{aligned}$$

for all $n \in \mathbb{N}$, where $\theta^n = (\theta^n(0, |D(x_0, x_1)|) \vee \theta^n(0, |D(x_0, x_0)|))$ for all $n \in \mathbb{N}$. This implies that for some $\eta \in \mathbb{N}$, we have

$$\begin{aligned} D(x_n, x_{n+\eta}) - D(x_n, x_n) &\leq D(x_n, x_{n+1}) + D(x_{n+1}x_{n+2}) + \dots \\ &+ D(x_{n+\eta-1}, x_{n+\eta}) - \sum_{i=0}^{\eta-1} D(x_{n+i}, x_{n+i}), \\ &\leq (\theta^n + \theta^{n+1} + \dots + \theta^{n+\eta-1})[|D(x_0, x_1)| + |D(x_0, x_0)|], \\ &\leq \frac{\theta^n}{1-\theta}[|D(x_0, x_1)| + |D(x_0, x_0)|]. \end{aligned}$$

for all $n \in \mathbb{N}$. Similarly we can calculate that

$$D(x_n, x_{n+\eta}) - D(x_{n+\eta}, x_{n+\eta}) \leq \frac{\theta^n}{1-\theta}[|D(x_0, x_1)| + |D(x_0, x_0)|].$$

It follows that $\lim_{n\to\infty} d_D^s(x_n, x_{n+\eta}) = 0$, thus, $\{x_n\}$ is a Cauchy sequence in (X, d_D^s). By Lemma 1.1 (X, d_D^s) is a complete metric space, therefore, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$ $i.e$ $\lim_{n\to\infty} d_D^s(x_n, x^*) = 0$ which then by Lemma 1.1 implies

$$D(x^*, x^*) = \lim_{n\longrightarrow\infty} D(x_n, x^*) = \lim_{n,m\longrightarrow\infty} D(x_n, x_m).$$

Now since $\lim_{n,m\to\infty}[D(x_n, x_m) - D(x_n, x_n)] = \lim_{n\to\infty} d_D(x_n, x_{n+k}) = 0$ which implies $\lim_{n,m\to\infty} D(x_n, x_m) = \lim_{n\to\infty} D(x_n, x_n) = 0$. It follows directly that

$$D(x^*, x^*) = \lim_{n\longrightarrow\infty} D(x_n, x^*) = 0.$$

The fact that $0 = D(x^*, x^*)$ yields that $0 \leq D(x^*, T(x^*))$, since $0 = D(x^*, x^*) \leq D(x^*, T(x^*))$. Moreover, for all $n \in \mathbb{N}$

$$\begin{aligned} D(x^*, T(x^*)) &\leq D(x^*, x_n) + D(x_n, T(x^*)) - D(x_n, x_n) \\ &\leq D(x^*, x_n) + |D(x_n, T(x^*))| + |D(x_n, x_n)| \\ &\leq D(x^*, x_n) + \alpha(x_{n-1}, x^*)|D(x_{n-1}, x^*)| + |D(x_n, x_n)| \\ &\leq D(x^*, x_n) + |D(x_{n-1}, x^*)| + |D(x_n, x_n)|. \end{aligned}$$

Hence we obtain that $D(x^*, T(x^*)) = 0$, since $\lim_{n\longrightarrow\infty} D(x_n, x_n) = \lim_{n\longrightarrow\infty} D(x_n, x^*) = 0$. We deduce that

$$D(x^*, Tx^*) = D(x^*, x^*) = D(T(x^*), T(x^*)) = 0.$$

This implies that $x^* = T(x^*)$, hence, x^* is a fixed point of T. Moreover, we have shown that $\{T^n(x)\}_{n\in\mathbb{N}}$ converges to x^* with respect to $\tau(d_D^s)$, for every $x \in X$.

It remains to prove the uniqueness of the fixed point. To this end, let us assume that $z \in X$ is a fixed point of T such that $z \neq x^*$. Then we have that $D(x^*, z) \neq 0$. In addition, we have that $D(x^*, z) > 0$, since $D(x^*, z) \geq D(x^*, x^*) \geq 0$. Furthermore,

$$D(x^*, z) = |D(x^*, z)| = |D(T(x^*), T(z))| \leq \alpha(x^*, z)|D(x^*, z)|.$$

It follows that $1 \leq \alpha(x^*, z) < 1$, which is a contradiction, therefore $x^* = z$. This completes the proof. $\square$

Note that Theorem 1.5 can be viewed as a particular case of Theorem 2.2.
A natural question that can be raised is whether the contractive condition of Theorem 2.2 can be replaced by the contractive condition in Theorem 1.5. The following easy example provides a negative answer to such a question.

Example 2.4. *Consider the complete dual partial metric* $(\mathbb{R}, D_\vee)$ *and the self-mapping* T_0 *defined as in Example 2.2. Then, fixed* $k \in [0,1[$, *it is not hard to verify that*

$$D_\vee(T(x), T(y)) \leq \alpha(x,y) D_\vee(x,y)$$

for all $x, y \in \mathbb{R}$ *with* $\alpha(x,y) = k$ *for all* $x, y \in \mathbb{R}$. *However,* T *has not fixed points. Observe that* T *does not hold the contractive condition of Theorem 2.2. Indeed, there is no mapping* $\alpha : \mathbb{R} \times \mathbb{R} \to [0,1[$ *such that*

$$1 = |D_\vee(-1,-1)| = |D_\vee(T_0(0), T_0(0))| \leq \alpha(0,0) D_\vee(0,0) = 0.$$

3. Conclusion

In this paper, fixed point theorems for Kannan contractive conditions and weakly contractive mappings in complete dual partial metric spaces have been discussed. Our analysis is based on the simple observation that fixed point results can be deduced from the metric in which distance between two points may be negative. We think that this aspect of finding the fixed points were overlooked and our paper will bring a lot of interest into this area. The significance of the above results lies in the fact that these results are true for all real numbers whereas such results proved in partial metric spaces are only true for positive real numbers.

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